A mathematical base for Fibre bundle formulation of Lagrangian quantum field theory

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Abstract

The paper contains a differential-geometric foundations for an attempt to formulate Lagrangian (canonical) quantum field theory on fibre bundles. In it the standard Hilbert space of quantum field theory is replace with a Hilbert bundle; the former playing a role of a (typical) fibre of the letter one. Suitable sections of that bundle replace the ordinary state vectors and the operators on the system's Hilbert space are transformed into morphisms of the same bundle. In particular, the field operators are mapped into corresponding field morphisms.

1. Introduction

The purpose of this work is to be presented grounds for a consistent formulation of quantum field theory in terms of fibre bundles. The ideas for that goal are shared from [1–6], where the quantum mechanics is formulated on the geometrical language of fibre bundle theory.

In section 2 contains some basic definitions. Special attention is paid on the Hilbert bundles, which will replace the Hilbert spaces of the ordinary quantum field theory, and the metric structure in them. In section 3 is considered an isomorphism between the fibres of a Hilbert bundle, called the bundle transport. It will play a central role in this investigation. Sect. 4 contains a motivation why the (Hilbert) fibre bundles are a natural scene for a mathematical formulation of quantum field theory. Sect. 5 concludes the paper.

2. Fibre Bundles. Hilbert bundles

To begin with, we present some facts from the theory of fibre bundles [7,8], in particular the Hilbert ones which will replace the Hilbert spaces in ordinary quantum field theory.

A *bundle* is a triple (E,π,B) of sets E and B, called (total) bundle space and base (space) respectively, and (generally) surjective mapping $\pi\colon E\to B$, called projection. If $b\in B$, $\pi^{-1}(b)$ is the fibre over b and, if $Q\subseteq B$, $(E,\pi,B)|_Q := (\pi^{-1}(Q),\pi|_{\pi^{-1}(Q)},Q)$ is the restriction on Q of a bundle (E,π,B). A *section* of (E,π,B) is a mapping $\sigma\colon B\to E$ such that $\pi\circ\sigma=\mathsf{id}_B$, where id_Z is the identity mapping of a set Z, and their set is denoted by $\mathrm{Sec}(E,\pi,B)$. The set of morphisms of (E,π,B) is

$$\mathrm{Mor}(E,\pi,B) := \{(\varphi,f)|\varphi\colon E\to E,\ f\colon B\to B,\ \pi\circ\varphi=f\circ\pi\}.$$

The set of all B-morphisms (strong morphisms) of (E,π,B) is $\mathrm{Mor}_B(E,\pi,B) := \{\varphi|\varphi\colon E\to E,\ \pi\circ\varphi=\pi\}$. Consider the set of *point-restricted morphisms*

$$\begin{aligned} E_0 :&= \{(\varphi_b,f)\,|\,\varphi_b=\varphi|_{\pi^{-1}(b)},\ b\in B,\ (\varphi,f)\in\mathrm{Mor}(E,\pi,B)\}\\ &= \{(\varphi_b,f)\,|\,\varphi_b\colon\pi^{-1}(b)\to\pi^{-1}(f(b)),\ b\in B,\ f\colon B\to B\}.\end{aligned}$$

Defining $\pi_0\colon E_0\to B$ by $\pi_0(\varphi_b,f):=b$ for $(\varphi_b,f)\in E_0$, we see that $\mathrm{mor}(E,\pi,B) := (E_0,\pi_0,B)$ is a fibre bundle. This is the *bundle of point-restricted morphisms* of (E,π,B). [1]

The *bundle* $\mathrm{mor}_B(E,\pi,B)$ *of point-restricted morphisms over* B of (E,π,B) has a base B, bundle space

$$\begin{aligned} E_0^B :&= \{\varphi_b\,|\,\varphi_b=\varphi|_{\pi^{-1}(b)},\ b\in B,\ \varphi\in\mathrm{Mor}_B(E,\pi,B)\}\\ &= \{\varphi_b\,|\,\varphi_b\colon\pi^{-1}(b)\to\pi^{-1}(b),\ b\in B\}\end{aligned}$$

and projection $\pi_0^B\colon E_0^B\to B$ such that $\pi_0^B(\varphi_b):=b,\quad \varphi_b\in E_0^B$. The bundle $\mathrm{mor}_B(E,\pi,B)$ will be refereed as the *bundle of restricted morphisms* of (E,π,B). There is a bijection $\mathrm{Mor}_B(E,\pi,B)\xrightarrow{\chi}\mathrm{Sec}\big(\mathrm{mor}_B(E,\pi,B)\big)$ given by $\chi\colon\varphi\mapsto\chi_\varphi$, $\varphi\in\mathrm{Mor}_B(E,\pi,B)$, with $\chi_\varphi\colon b\mapsto\chi_\varphi(b):=\varphi|_{\pi^{-1}(b)}$, $b\in B$. Its inverse is $\chi^{-1}\colon\sigma\mapsto\chi^{-1}(\sigma)=\varphi$, $\sigma\in\mathrm{Sec}(\mathrm{mor}_B(E,\pi,B))$, with $\varphi\colon E\to E$ given via $\varphi|_{\pi^{-1}(b)}=\sigma(b)$ for every $b\in B$.

A mapping $\mathrm{Sec}(E,\pi,B)\to\mathrm{Sec}(E,\pi,B)$ will be called *morphism of the set* $\mathrm{Sec}(E,\pi,B)$ *of sections of bundle* (E,π,B). The set of all such mappings will be denoted by $\mathrm{MorSec}(E,\pi,B)$.

There is a natural mapping from $\mathrm{Mor}_B(E,\pi,B)$ into $\mathrm{MorSec}(E,\pi,B)$ given for $A\in\mathrm{Mor}_B(E,\pi,B)$ and $X\in\mathrm{Sec}(E,\pi,B)$ by $\hat{A}\in\mathrm{MorSec}(E,\pi,B)$ by $\hat{A}(X):=A\circ X$. We say that $\hat{A}$ as a *morphism of sections (of* (E,π,B)*) induced or generated by* A.

When *vector bundles* are considered, in the definition of a morphism or B-morphism is included the condition that the corresponding fibre mappings must be *linear*.

[1] There exists a bijective correspondence ρ such that $\mathrm{Mor}(E,\pi,B)\xrightarrow{\rho}\mathrm{Sec}\big(\mathrm{mor}(E,\pi,B)\big)$.

Definition 2.1. A *Hilbert (fibre) bundle* is a vector bundle whose fibres over the base are isomorphic Hilbert spaces or, equivalently, whose (standard) fibre is a Hilbert space.

Some quite general aspects of the Hilbert bundles can be found in [9, chapter VII].

Let $(F,\pi,M,\mathcal{F})$ be a Hilbert bundle with bundle space F, base M, projection π, and (typical) fibre $\mathcal{F}$. The fibre over $x\in M$ will be often denoted by F_x, $F_x:=\pi^{-1}(x)$. Let $l_x\colon F_x\to\mathcal{F}$, $x\in M$, be the isomorphisms defined by the restricted decomposition functions, viz., as $\phi_W|_x\colon \{x\}\times\mathcal{F}\to F_x$ with W being a neighborhood of x, we define l_x via $\phi_W|_x(x,\psi)=: l_x^{-1}(\psi)\in\pi^{-1}(x)$ for every $\psi\in\mathcal{F}$ and call l_x *point-trivializing mappings (isomorphisms).*

Let $\prec\cdot|\cdot\succ\colon\mathcal{F}\times\mathcal{F}\to\mathbb{R}$ be the scalar product in the Hilbert space $\mathcal{F}$ and for every $x\in M$ the mapping $\langle\cdot|\cdot\rangle_x\colon F_x\times F_x\to\mathbb{R}$ be the scalar product in the fibre F_x considered as a Hilbert space. The vector structure of $(F,\pi,M,\mathcal{F})$ is called *compatible* with its metric structure if the isomorphisms $l_x\colon F_x\to\mathcal{F}$ preserve the scalar products, viz. iff $\langle\varphi_x|\psi_x\rangle_x=\prec l_x(\varphi_x)|l_x(\psi_x)\succ$ for every $\varphi_x,\psi_x\in F_x$. A Hilbert bundle with compatible vector and metric structure will be called *compatible Hilbert bundle.* In such a bundle the isomorphisms l_x, $x\in M$ transform the metric structure $\prec\cdot|\cdot\succ$ from $\mathcal{F}$ to F and v.v. according to

$$\langle\cdot|\cdot\rangle_x=\prec l_x\,\cdot\,|l_x\cdot\succ,\qquad x\in M \tag{2.1}$$

$$\prec\cdot|\cdot\succ=\langle l_x^{-1}\,\cdot\,|l_x^{-1}\cdot\rangle_x,\qquad x\in M. \tag{2.1$'$}$$

The mappings $l_{x\to y}:=l_y^{-1}\circ l_x\colon\pi^{-1}(x)\to\pi^{-1}(y)$ are (i) fibre mappings for fixed y, (ii) linear isomorphisms, and (iii) isometric, i.e. metric preserving in a sense that

$$\langle l_{x\to y}\,\cdot\,|l_{x\to y}\cdot\rangle_y=\langle\cdot|\cdot\rangle_x. \tag{2.2}$$

Consequently, all of the fibres over the base and the standard fibre of a compatible Hilbert bundle are (linearly) isometric and isomorphic Hilbert spaces.

Beginning from now on in the present investigation, *only compatible Hilbert bundles will be employed.* For brevity, we shall call them simply Hilbert bundles.

Defining the *Hermitian conjugate* mapping $A_x^{\ddagger}\colon\mathcal{F}\to F_x$ of a mapping $A_x\colon F_x\to\mathcal{F}$ by

$$\langle A_x^{\ddagger}\varphi|\chi_x\rangle_x:=\prec\varphi|A_x\chi_x\succ,\qquad \varphi\in\mathcal{F},\quad \chi_x\in F_x, \tag{2.3}$$

we find (see (2.1), the dagger denotes Hermitian conjugation in $\mathcal{F}$)

$$A_x^{\ddagger}=l_x^{-1}\circ\big(A_x\circ l_x^{-1}\big)^{\dagger}. \tag{2.4}$$

We call a mapping $A_x\colon F_x\to\mathcal{F}$ *unitary* if it has an inverse A^{-1} and

$$A_x^{\ddagger}=A_x^{-1}. \tag{2.5}$$

Evidently, the isometric isomorphisms $l_x\colon F_x\to\mathcal{F}$ are unitary in this sense:

$$l_x^{\ddagger}=l_x^{-1}. \tag{2.6}$$

The *Hermitian conjugate* mapping to a mapping $A_{x\to y}$ in a set $\{C_{x\to y}\colon F_x\to F_y,\ x,y\in M\}$ of mappings between the fibres of the bundle is a mapping $A_{x\to y}^{\ddagger}\colon F_x\to F_y$ defined via

$$\langle A_{x\to y}^{\ddagger}\varPhi_x|\varPsi_y\rangle_y:=\langle\varPhi_x|A_{y\to x}\varPsi_y\rangle_x,\qquad \varPhi_x\in F_x,\quad \varPsi_y\in F_y. \tag{2.7}$$

Its explicit form is

$$A_{x\to y}^{\ddagger}=l_y^{-1}\circ\big(l_x\circ A_{y\to x}\circ l_y^{-1}\big)^{\dagger}\circ l_x. \tag{2.8}$$

As $(\mathcal{A}^{\dagger})^{\dagger}\equiv\mathcal{A}$ for any $\mathcal{A}\colon\mathcal{F}\to\mathcal{F}$, we have

$$\big(A_{x\to y}^{\ddagger}\big)^{\ddagger}=A_{x\to y}. \tag{2.9}$$

If $B_{x\to y} \in \{C_{x\to y} \colon F_x \to F_y,\ x, y \in M\}$, then a simple verification shows

$$(B_{y\to z} \circ A_{x\to y})^{\ddagger} = A^{\ddagger}_{y\to z} \circ B^{\ddagger}_{x\to y}, \qquad x, y, z \in M. \tag{2.10}$$

A mapping $A_{x\to y}$ is called *Hermitian* if

$$A^{\ddagger}_{x\to y} = A_{x\to y}. \tag{2.11}$$

A calculation proves that the mappings $l_{x\to y} := l_y^{-1} \circ l_x$ are Hermitian,

$$(l_{x\to y})^{\ddagger} = l_{x\to y}. \tag{2.12}$$

A mapping $A_{x\to y} \colon F_x \to F_y$ is called *unitary* if it has a left inverse mapping and

$$A^{\ddagger}_{x\to y} = A^{-1}_{y\to x}, \tag{2.13}$$

where $A^{-1}_{x\to y} \colon F_y \to F_x$ is the *left* inverse of $A_{x\to y}$, i.e. $A^{-1}_{x\to y} \circ A_{x\to y} := \mathsf{id}_{F_x}$.

A simple verification by means of (2.7) shows the equivalence of (2.13) with

$$\langle A_{y\to x} \,\cdot\, | A_{y\to x} \,\cdot\, \rangle_x = \langle \cdot | \cdot \rangle_y \colon F_y \times F_y \to \mathbb{C}, \tag{2.13$'$}$$

i.e. the unitary mappings are fibre-metric compatible in a sense that they preserve the fibre scalar (inner) product. Such mappings will be called *fibre-isometric* or simply *isometric*.

It is almost evident that the mappings

$$l_{x\to y} := l_y^{-1} \circ l_x \colon \pi^{-1}(x) \to \pi^{-1}(y) \tag{2.14}$$

are unitary, that is we have:[2]

$$l^{\ddagger}_{x\to y} = l_{x\to y} = l^{-1}_{y\to x}. \tag{2.15}$$

Let A be a morphism over M of $(F, \pi, M, \mathcal{F})$, i.e. $A \colon F \to F$ and $\pi \circ A = \pi$, and $A_x := A|_{F_x}$. The *Hermitian conjugate* bundle morphism $A^{\ddagger}$ to A is defined by (cf. (2.7))

$$\langle A^{\ddagger} \varPhi_x | \varPsi_x \rangle_x := \langle \varPhi_x | A \varPsi_x \rangle_x, \qquad \varPhi_x,\, \varPsi_x \in F_x. \tag{2.16}$$

Thus (cf. (2.8))

$$A^{\ddagger}_x := A^{\ddagger}|_{F_x} = l_x^{-1} \circ \big(l_x \circ A_x \circ l_x^{-1}\big)^{\dagger} \circ l_x. \tag{2.17}$$

A bundle morphism A is called *Hermitian* if $A^{\ddagger}_x = A_x$ for every $x \in M$, i.e. if

$$A^{\ddagger} = A, \tag{2.18}$$

and it is called *unitary* if $A^{\ddagger}_x = A_x^{-1}$ for every $x \in M$, i.e. if

$$A^{\ddagger} = A^{-1}. \tag{2.19}$$

Using (2.16), we can establish the equivalence of (2.18) and

$$\langle A \,\cdot\, | A \,\cdot\, \rangle_x = \langle \cdot | \cdot \rangle_x \colon F_x \times F_x \to \mathbb{C}. \tag{2.20}$$

Consequently the unitary morphisms are fibre-metric compatible, i.e. they are *isometric* in a sense that they preserve the fibre Hermitian scalar (inner) product.

To deal with the differentiable properties of the employed Hilbert bundle $(F, \pi, M, \mathcal{F})$, we will require the bundle space F and the base space M be of class C^1. Moreover, at the present level of development of quantum field theory, we can identify the base M with the 4-dimensional Minkowski space-time. We shall require also the point-trivializing isomorphisms l_x to have a C^1 dependence on $x \in M$, i.e. the mapping $l \colon F \to \mathcal{F}$ given by $l \colon u \mapsto l_{\pi(u)} u$ for $u \in F$, to be of class C^1 as a mapping between manifolds. A Hilbert bundle with the last property will be called C^1 bundle (or bundle of class C^1).

[2] The Hermiticity and at the same time unitarity of $l_{x\to y}$ is not incidental as they define a (flat) linear transport (along paths or along the identity mapping of M) in $(F, \pi, M, \mathcal{F})$. For details, see [1, 6].

3. The bundle transport

Suppose $(E, \pi, B, \mathcal{E})$ is a $\mathbb{K}$-vector bundle, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, such that E, B, and $\mathcal{E}$ are C^1 manifolds and its point-trivializing isomorphisms l_x, $x \in B$, are of class C^1, i.e. it is of class C^1.

Definition 3.1. The *bundle transport* in a bundle $(E, \pi, B, \mathcal{E})$ is a mapping $l\colon (x, y) \mapsto l_{x\to y}$, $x, y \in B$, where the mapping

$$l_{x\to y}\colon \pi^{-1}(x) \to \pi^{-1}(y), \tag{3.1}$$

called *(bundle) transport from x to y*, is defined by

$$l_{x\to y} := l_y^{-1} \circ l_x \tag{3.2}$$

with $l_x\colon \pi^{-1}(x) \to \mathcal{E}$, $x \in B$, being the point-trivializing isomorphisms of $(E, \pi, B, \mathcal{E})$.

Here are some frequently used properties of the bundle transport which are consequences of (3.2) and the linearity of the isomorphisms l_x, $x \in B$:

$$l_{x\to y} \circ l_{z\to x} = l_{z\to y}, \qquad x, y, z \in B, \tag{3.3}$$

$$l_{x\to x} = \mathsf{id}_{\pi^{-1}(x)}, \qquad x \in B, \tag{3.4}$$

$$l_{x\to y}(\lambda u + \mu v) = \lambda l_{x\to y} u + \mu l_{x\to y} v, \qquad \lambda, \mu \in \mathbb{K}, \quad u, v \in \pi^{-1}(x), \tag{3.5}$$

$$(l_{x\to y})^{-1} = l_{y\to x}. \qquad x, y \in B. \tag{3.6}$$

The bundle transport is Hermitian and unitary in a sense that such are $l_{x\to y}$, $x, y \in B$.

A section $X \in \mathrm{Sec}(E, \pi, B, \mathcal{E})$ is called *l-transported* (or (l-)constant) if for some (and hence any) $x \in B$ and every $y \in B$ is fulfilled

$$X(y) = l_{x\to y}(X(x)) \tag{3.7}$$

where $X(x) := X_x$ with $X\colon x \mapsto X_x$. Such a section is uniquely defined by specifying its value at a single point. If $\mathcal{X}_0$ is a fixed vector in the fibre $\mathcal{E}$, the section X given via

$$X(x) = l_x^{-1}(\mathcal{X}_0) \tag{3.8}$$

is l-transported. Such sections will represent the states of quantum fields in our approach.

Let (U, κ), $\kappa\colon U \to \mathbb{R}^{\dim B}$, be a local chart in a neighborhood U of $x \in B$ and $x(\varepsilon, \mu) \in U$, where ε is a real number in a neighborhood of the zero and $\mu = 1, \dots, \dim B$, has coordinates $\kappa^\nu(x(\varepsilon, \mu)) = \kappa^\nu(x) + \varepsilon\delta_\mu^\nu$ with $\delta_\mu^\nu = 1$ for $\nu = \mu$ and $\delta_\mu^\nu = 0$ for $\nu \neq \mu$. According to the general formalism (see [10]) in any κ the bundle transport generates derivations

$$D_\mu\colon\ \mathrm{Sec}^1(E, \pi, B) \to \mathrm{Sec}^0(E, \pi, B) \qquad \mu = 1, \dots, \dim B \tag{3.9}$$

defined via their action on a C^1 section Y by

$$(D_\mu Y)(x) := D_\mu\big|_x(Y) := \lim_{\varepsilon\to 0} \frac{l_{x(\varepsilon,\mu)\to x}\big(Y(x(\varepsilon, \mu))\big) - Y(x)}{\varepsilon}. \tag{3.10}$$

The bundle transport also generates the mappings (derivations)

$$\begin{aligned} \hat{D}_\mu\colon &\big\{\hat{A} \in \mathrm{MorSec}^1(E, \pi, B) \text{ is of class } C^1 \text{ and } \hat{A}(\,\cdot\,) = A \circ (\,\cdot\,) \text{ with } A \in \mathrm{Mor}^1_B(E, \pi, B)\big\} \\ &\to \big\{\hat{A} \in \mathrm{MorSec}^0(E, \pi, B) \text{ is of class } C^0 \text{ and } \hat{A}(\,\cdot\,) = A \circ (\,\cdot\,) \text{ with } A \in \mathrm{Mor}^0_B(E, \pi, B)\big\}, \end{aligned} \tag{3.11}$$

$$\hat{D}_\mu(\hat{A}) := [D_\mu, \hat{A}]_{\_} = D_\mu \circ \hat{A} - \hat{A} \circ D_\mu \tag{3.12}$$

for any C^1 morphism $\hat{A}$ of $\mathrm{Sec}^1(E,\pi,B)$ generated by a morphisms $A\in\mathrm{Mor}^1_B(E,\pi,B)$.

The mappings D_μ and $\hat{D}_\mu$ are derivations in a sense that they are *linear* and

$$D_\mu(f\,Y)=\frac{\partial f}{\partial x^\mu}\,Y+f\,D_\mu(Y), \tag{3.13}$$
$$\hat{D}_\mu(\hat{A}\circ\hat{C})=(\hat{D}_\mu(\hat{A}))\circ\hat{C}+\hat{A}\circ(\hat{D}_\mu(\hat{C})) \tag{3.14}$$

where f is a C^1 function on B and $\hat{C}$ and $\hat{A}$ are C^1 morphisms of $\mathrm{Sec}^1(E,\pi,B)$.

Proposition 3.1. *If Y is a C^1 section, A a C^1 morphism of (E,π,B), and $\hat{A}(\,\cdot\,)=A\circ(\,\cdot\,)$ is the morphisms of* $\mathrm{Sec}(E,\pi,B)$ *generated by A, then*

$$(D_\mu Y)(x)=l_x^{-1}\Big(\frac{\partial(l_x(Y(x)))}{\partial x^\mu}\Big)=\big(l_x^{-1}\circ\frac{\partial}{\partial x^\mu}\circ l_x\big)(Y(x)), \tag{3.15}$$
$$\big((\hat{D}_\mu\hat{A})(Y)\big)(x)=\Big(l_x^{-1}\circ\frac{\partial(l_x\circ A_x\circ l_x^{-1})}{\partial x^\mu}\circ l_x\Big)(Y(x)). \tag{3.16}$$

Proof. First of all, note that since $l_x(Y(x))$ is a vector in the standard fibre $\mathcal{E}$ and $l_x\circ A\circ l_x^{-1}=l_x\circ A|_{\pi^{-1}(x)}\circ l_x^{-1}$ is an operator in it, the partial derivatives of them are well defined. The equality (3.15) is a simple consequence of (3.10) and (3.2). To verify (3.16) one should apply (3.12) to a C^1 section Y, evaluate the result at x, and next to use (3.15). □

If $\{x^\mu\}\mapsto\{x'^\mu\}$ is a change of the local coordinates, then D_μ and $\hat{D}_\mu$ behave like basic (tangent) vector fields over B:

$$D_\mu\mapsto D'_\mu=\frac{\partial x^\nu}{\partial x'^\mu}D_\nu,\quad \hat{D}_\mu\mapsto\hat{D}'_\mu=\frac{\partial x^\nu}{\partial x'^\mu}\hat{D}_\nu. \tag{3.17}$$

This result, which follows from proposition 3.1, shows that one can define invariant 'covariant derivatives' $D_V:=V^\mu D_\mu$ and $\hat{D}_V:=V^\mu\hat{D}_\mu$ along a vector field $V=V^\mu\frac{\partial}{\partial x^\mu}$ tangent to B.

The bundle transport l induces a natural '*transport along the identity mapping of B*' in the bundle of point-restricted morphisms $\mathrm{mor}_B(E,\pi,B)=(E_0,\pi_0,B)$. By this term we denote a mapping ${}^\circ l\colon(x,y)\mapsto{}^\circ l_{x\to y}$, $x,y\in B$, where the mapping ${}^\circ l_{x\to y}\colon\pi_0^{-1}(x)\to\pi_0^{-1}(y)$, called *transport from x to y in* $\mathrm{mor}_B(E,\pi,B)$, is defined by

$${}^\circ l_{x\to y}(\chi_x):=l_{x\to y}\circ\chi_x\circ l_{y\to x}=l_y^{-1}\circ(l_x\circ\chi_x\circ l_x^{-1})\circ l_y\colon\pi^{-1}(y)\to\pi^{-1}(y) \tag{3.18}$$

where $\chi_x\colon\pi^{-1}(x)\to\pi^{-1}(x)$. One can check that the transports ${}^\circ l_{x\to y}$ are $\mathbb{K}$-*linear* and satisfy the basic 'transport equations' (cf. (3.3)–(3.6)):

$${}^\circ l_{x\to y}\circ{}^\circ l_{z\to x}={}^\circ l_{z\to y},\qquad x,y,z\in B, \tag{3.19}$$
$${}^\circ l_{x\to x}=\mathsf{id}_{\{\pi^{-1}(x)\to\pi^{-1}(x)\}},\qquad x\in B, \tag{3.20}$$
$$\big({}^\circ l_{x\to y}\big)^{-1}={}^\circ l_{y\to x},\qquad x,y\in B. \tag{3.21}$$

We shall call ${}^\circ l$ the *transport associated to the bundle transport l* (in $\mathrm{mor}_B(E,\pi,B)$).

A morphism $A\in\mathrm{Mor}_B(E,\pi,B)$ is said to be ${}^\circ l$-transported (or (${}^\circ l$-) constant) if the restrictions $A_x:=A|_{\pi^{-1}(x)}$ satisfy the equation

$$A_y={}^\circ l_{x\to y}(A_x)=l_{x\to y}\circ A_x\circ l_{y\to x},\qquad x,y\in B. \tag{3.22}$$

One can construct such kind of morphisms in the following way. Let us fix $x\in B$ and $\chi_0\colon\pi^{-1}(x)\to\pi^{-1}(x)$. The morphism ${}^\circ l_x(\chi_0)\in\mathrm{Mor}_B(E,\pi,B)$ such that

$${}^\circ l_x(\chi_0)|_{\pi^{-1}(y)}:={}^\circ l_{x\to y}(\chi_0)=l_{x\to y}\circ\chi_0\circ l_{y\to x} \tag{3.23}$$

is ${}^\circ l$-transported due to (3.19). Analogously, for fixed x, any morphism $A \in \mathrm{Mor}_B(E,\pi,B)$ defines an ${}^\circ l$-transported morphism, denoted by ${}^\circ l_x(A)$ and such that

$$ {}^\circ l_x(A) := {}^\circ l_x(A_x) \colon y \mapsto ({}^\circ l_x A)_y := {}^\circ l_{x\to y} A_x = l_{x\to y} \circ A_x \circ l_{y\to x}. \tag{3.24}$$

Similarly to the transport l considered above, the transport ${}^\circ l$ induces derivations ${}^\circ D_\mu$ on the set $\mathrm{Sec}^1(\mathrm{mor}_B(E,\pi,B))$ according to

$$ ({}^\circ D_\mu A)(x) := \lim_{\varepsilon\to 0} \frac{{}^\circ l_{x(\varepsilon,\mu)\to x}\bigl(A_{x(\varepsilon,\mu)}\bigr) - A_x}{\varepsilon} \tag{3.25}$$

with $x(\varepsilon,\mu)$ defined above and $A \in \mathrm{Sec}^1(\mathrm{mor}_B(E,\pi,B))$. Applying (3.18), we find

$$ ({}^\circ D_\mu A)(x) = l_x^{-1} \circ \frac{\partial (l_x \circ A(x) \circ l_x^{-1})}{\partial x^\mu} \circ l_x. \tag{3.26}$$

If $\chi\colon \mathrm{Mor}_B(E,\pi,B) \to \mathrm{Sec}(\mathrm{mor}_B(E,\pi,B))$ is the bijection mentioned in Subsect. 2, $A \in \mathrm{Mor}_B(E,\pi,B)$, and $\hat{A}(\,\cdot\,) = A \circ (\,\cdot\,)$, then (see (3.16))

$$ \bigl((\hat{D}_\mu(\hat{A}))(Y)\bigr)(x) = \bigl({}^\circ D_\mu(\chi(A))\bigr)(Y(x)). \tag{3.27}$$

If, for every morphism $\hat{A} \in \mathrm{MorSec}(F,\pi,M,\mathcal{F})$, we define a mapping $\breve{l}_x\hat{A}$ from the set $\mathrm{Sec}^1(F,\pi,M,\mathcal{F})$ to the set of mappings from M into $\pi^{-1}(x)$ by

$$ ((\breve{l}_x\hat{A})(Y))(y) := (l_{y\to x}\circ \hat{A}_y)(Y(y)) - (\hat{A}_x \circ (l_{y\to x})(Y(y)), \tag{3.28}$$

with $y \in M$ and $Y \in \mathrm{Sec}(F,\pi,M,\mathcal{F})$, we have the relation

$$ ((\hat{D}_\mu\hat{A})(Y))(x) = \Bigl\{ \frac{\partial}{\partial y^\mu} [((\breve{l}_x\hat{A})(Y))(y)] \Bigr\}\Bigr|_{y=x}. \tag{3.29}$$

To prove it, one should insert (3.2) into (3.28), to apply to the obtained equality the operator $\frac{\partial}{\partial y^\mu}$, then to put $y = x$ and to use (3.16).

Now we shall present some expressions in (local) bases which may be a little more familiar to the physicists. Sums like $a^i b_i$ should be understood as a sum over i and/or integral over i if the index i takes countable and/or uncountable values; for some more details — see [2].

Let $\{e_i(x)\}$ be a basis in $\pi^{-1}(x)$, $\{f_i\}$ a basis in $\mathcal{F}$ with $\partial_\mu f_i = 0$, and $\boldsymbol{l}_x$ and $\boldsymbol{l}(y,x)$ be the matrices of respectively l_x and $l_{x\to y}$ in them, i.e. if $l_x(e_i(x)) =: {l_x}^j{}_i f_j$ and $l_{x\to y}(e_i(x)) =: l^j{}_i(y,x) e_i(y)$. Then $\boldsymbol{l}_x := [{l_x}^j{}_i]$ and $\boldsymbol{l}(y,x) := [l^j{}_i(y,x)] = \boldsymbol{l}^{-1}(y)\boldsymbol{l}(x)$ (see (3.2).) Then, ,for $Y = Y^i e_i$ from (3.10), we get (cf. [10])[3]

$$ (D_\mu Y)(x) = \Bigl(\frac{\partial Y^i(x)}{\partial x^\mu} + \Gamma^i{}_{j\mu}(x) Y^j(x) \Bigr) e_i(x) \tag{3.30}$$

$$ (D_\mu e_j)(x) =: \Gamma^i{}_{j\mu}(x) e_i(x) \tag{3.31}$$

with

$$ \Gamma_\mu(x) := [\Gamma^i{}_{j\mu}(x)] = \frac{\partial \boldsymbol{l}(y,x)}{\partial x^\mu}\Bigr|_{y=x} = \boldsymbol{l}_x^{-1} \frac{\partial \boldsymbol{l}(x)}{\partial x^\mu} \tag{3.32}$$

beingare the matrices of the *coefficients* (resp. *components*) $\Gamma^i{}_{j\mu}$ of l (resp. D_μ).

Under the changes $e_i \mapsto e'_i = C^j_i e_j$ and $x^\mu \mapsto x'^\mu$, $C = [C^j_i]$ being non-degenerate matrix-valued C^1 function, the matrices Γ_μ transform into [10, eq. (4.8)]

$$ \Gamma'_\mu(x) := \Bigl(C^{-1}(x)\Gamma_\nu(x)C(x) + C^{-1}(x)\frac{\partial C(x)}{\partial x^\nu} \Bigr) \frac{\partial x^\nu}{\partial x'^\mu}. \tag{3.33}$$

[3] In this paper we assume the Einstein summation convention: over indices repeated on different levels a summation over their whole range is implicitly understood.

Applying (3.12) to some C^1 section A and taking into account (3.30) and (3.32), we find the matrix of $\hat{D}_\mu \hat{A}$ in $\{e_i\}$ as

$$[(\hat{D}_\mu \hat{A})^j_i]|_x = \Big(\frac{\partial \boldsymbol{A}}{\partial x^\mu} + [\Gamma_\mu(x), \boldsymbol{A}]_{\_} \Big)\Big|_x = \boldsymbol{l}_x^{-1} \frac{\partial \boldsymbol{\mathcal{A}}(x)}{\partial x^\mu} \boldsymbol{l}_x \tag{3.34}$$

with $\boldsymbol{A}$ being the matrix of A in $\{e_i\}$. Here the second equality is a consequence of $\boldsymbol{A} = \boldsymbol{l}_x^{-1} \boldsymbol{\mathcal{A}} \boldsymbol{l}_x$ which is equivalent to $\hat{A}_x = l_x^{-1} \circ \mathcal{A}(x) \circ l_x$.

4. On bundle formulation of quantum field theory

A quantum field $\boldsymbol{\varphi}$ is a operator-valued vector distribution (generalized function), $\boldsymbol{\varphi} = (\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_n)$ with $n \in \mathbb{N}$ and $\boldsymbol{\varphi}_i$, $i = 1, \dots, n$, being operator-valued distributions called components of $\boldsymbol{\varphi}$, such that

$$\boldsymbol{\varphi}(f) = \int_M \mathrm{d}^4 y \sum_i \varphi_i(y) f^i(y) = \sum_i \boldsymbol{\varphi}_i(f^i), \quad \boldsymbol{\varphi}_i(g) := \int_M \mathrm{d}^4 y \varphi_i(y) g(y) \tag{4.1}$$

for a vector-valued test function $f = (f^1, \dots, f^n)$ and $f^i, g \colon M \to \mathbb{K}$ being test functions. In this (symbolic) equation $\varphi_i(y)$ are the components of the non-smeared field $\varphi = (\varphi_1, \dots, \varphi_n)$ which (as well as $\boldsymbol{\varphi}(f)$) are operators acting on the state vectors in the system's (field's) Hilbert space $\mathcal{F}$ of states. Here and below, we implicitly adopt the Heisenberg picture of motion.

The discussion in [1, subsect. 4.2] can *mutatis mutandis* be repeated here with respect to the state vectors. In short, this leads to the replacement of the system's Hilbert space $\mathcal{F}$ of states with a Hilbert fibre bundle $(F, \pi, M, \mathcal{F})$ (of states) with Minkowski spacetime M as a base, projection $\pi \colon F \to M$, fibres $\pi^{-1}(x)$ with $x \in M$, and the system's ordinary Hilbert space $\mathcal{F}$ as a (standard, typical) fibre. Let $\{l_x : x \in M\}$ be a set of (linear) isomorphisms $l_x \colon F_x \to \mathcal{F}$. Given an observer O_x at $x \in M$, if a system is characterized by a state vector $\mathcal{X} \in \mathcal{F}$, the vector

$$X(x) := l_x^{-1}(\mathcal{X}) \tag{4.2}$$

should be considered as its state vector relative to O_x. Hereof the state vector $\mathcal{X}$ is replace with a state section $X \in \operatorname{Sec}(F, \pi, M, \mathcal{F})$ such that $X \colon x \mapsto X(x) = l_x^{-1}(\mathcal{X})$.

Accepting the above bundle description of states, to the bundle description of quantum fields and, in general, to any operator $\mathcal{A}(x) \colon \mathcal{F} \to \mathcal{F}$ can be applied arguments similar to the ones in [2, subsect. 3.1]. At a point $x \in M$, the bundle analogue A_x of such a mapping $\mathcal{A}(x)$ can be defined by requiring the transition $\mathcal{A}(x) \mapsto A_x$ to preserve the scalar products,[4] i.e.

$$\prec \mathcal{X} | \mathcal{A}(x)(\mathcal{Y}) \succ = \langle X(x) | A_x(Y(x)) \rangle_x, \tag{4.3}$$

where $\mathcal{X}, \mathcal{Y} \in \mathcal{F}$, $x \in M$, $X(x) = l_x^{-1}(\mathcal{X})$, $Y(x) = l_x^{-1}(\mathcal{Y})$, and $\prec \cdot | \cdot \succ$ and $\langle \cdot | \cdot \rangle_x$ are the scalar products in the Hilbert spaces $\mathcal{F}$ and $\pi^{-1}(x)$, respectively, which are connected by (see Subsect. 3 or [1, eq. (3.1)])

$$\langle \cdot | \cdot \rangle_x = \prec l_x \cdot | l_x \cdot \succ, \qquad x \in M \tag{4.4}$$

$$\prec \cdot | \cdot \succ = \langle l_x^{-1} \cdot | l_x^{-1} \cdot \rangle_x, \qquad x \in M. \tag{4.4$'$}$$

In particular, if $\mathcal{A}(x)$ is an operator representing a dynamical variable $\mathbb{A}$, equation (4.3) implies that the observed (mean) value of $\mathbb{A}$ is independent of the way we compute (or describe) it. Combining (4.3) and (4.4$'$), we derive

$$A_x = l_x^{-1} \circ \mathcal{A}(x) \circ l_x. \tag{4.5}$$

[4] Physically this means independence of the observed values of the dynamical variables of the way we calculate them.

In this way, we see that to an *operator* $\mathcal{A}(x)\colon \mathcal{F} \to \mathcal{F}$ there corresponds a *morphism*

$$A \in \mathrm{Mor}_M(F, \pi, M, \mathcal{F}) \tag{4.6}$$

of the system's Hilbert bundle with $A|_{\pi^{-1}(x)} = A_x$ given via (4.5).

In particular, the non-smeared components φ_i of a quantum field $\boldsymbol{\varphi}$ change to $\Phi_i \in \mathrm{Mor}_M(F, \pi, M, \mathcal{F})$ with

$$\Phi_i|_{\pi^{-1}(x)} := l_x^{-1} \circ \varphi_i(x) \circ l_x. \tag{4.7}$$

Similarly, the smeared field components $\boldsymbol{\varphi}_i$ should be replaced with mappings

$$\boldsymbol{\Phi}_i\colon x \mapsto \boldsymbol{\Phi}_i|_x := l_x^{-1} \circ \boldsymbol{\varphi}_i(\,\cdot\,) \circ l_x. \tag{4.8}$$

as a result of which the smeared field $\boldsymbol{\Phi}$ becomes a mapping

$$\boldsymbol{\Phi}\colon x \mapsto \boldsymbol{\Phi}|_x = l_x^{-1} \circ \boldsymbol{\varphi}(\,\cdot\,) \circ l_x, \tag{4.9}$$

so that

$$\boldsymbol{\Phi}_x(f) = \int_M \mathrm{d}^4 y \sum_i {}^{\circ}l_{y\to x}(\Phi_i(y)) f^i(y) \tag{4.10}$$

where equations (4.1) and (4.7) were used and the transport ${}^{\circ}l$ in $\mathrm{mor}_M(F, \pi, M, \mathcal{F})$ is defined via (3.18).

However, the above-introduced morphisms, like A, are not the exact objects we need. Here are two reasons for this. On one hand, since in the ordinary theory the operators like, $\mathcal{A}$, act on state vectors, like $\mathcal{X}$, we should expect the bundle analogue $\hat{A}$ of $\mathcal{A}$ to act on a state section X producing again some section $\hat{A}(X) \in \mathrm{Sec}(F, \pi, M, \mathcal{F})$. On another hand, since in the module $\mathrm{Sec}(F, \pi, M, \mathcal{F})$ there is a natural scalar product $\langle\,\cdot\,|\,\cdot\,\rangle$ with values in the $\mathbb{C}$-valued functions (see Subsect. 3), the r.h.s. of (4.3) should be identified with the value at $x \in M$ of the scalar product $\langle X|\hat{A}(Y)\rangle$. Combining these ideas, we conclude that

$$\hat{A}(X) = A \circ X. \tag{4.11}$$

Hereof, the object $\hat{A}$ is the morphism in $\mathrm{MorSec}(F, \pi, M, \mathcal{F})$ generated by the morphism $A \in \mathrm{Mor}_M(F, \pi, M, \mathcal{F})$ whose restriction on $\pi^{-1}(x)$ is (4.5).

Assuming $\hat{A}$ to be the 'right' bundle analogue of $\mathcal{A}$, we conclude that

$$\hat{A}(X)\colon x \mapsto \hat{A}(X)|_x = (A \circ X)(x) = A_x(X(x)) = l_x^{-1}(\mathcal{A}(x)(\mathcal{X})), \tag{4.12}$$

i.e. the image of $\mathcal{A}(x)(\mathcal{X})$ according to (4.1) is exactly the value at x of $\hat{A}(X)$ and

$$\prec \mathcal{X}|\mathcal{A}(x)(\mathcal{Y})\succ = \langle X|\hat{A}(Y)\rangle|_x \tag{4.13}$$

which expresses the invariance of the scalar products when we replace state vectors with state sections. Since the scalar products define the observed (expectation, mean) values of physically observable quantities, the last equality expresses the independence of these values of the way we calculate them.

5. Conclusion

The present investigation can be regarded as a continuation of the fibre bundle formulation of quantum physics begun in [1–5]. Here we have applied a slightly different approach to the (canonical) quantum field theory in Heisenberg picture. The basic idea is the standard Hilbert space of states to be replaced with a Hilbert bundle with it as a (typical) fibre, then all quantities of the ordinary quantum field theory are mapped into their bundle analogues by means of the bundle transport, which is an internal object of any particular (locally trivial) bundle, or other mappings build from it.

However, the realization of such a procedure is not unique. The main reason being that the quantum field theory has not a unique generally accepted formulation.

References

[1] Bozhidar Z. Iliev. Fibre bundle formulation of nonrelativistic quantum mechanics. I. Introduction. The evolution transport. *Journal of Physics A: Mathematical and General*, 34(23):4887–4918, 2001. DOI No. 10.1088/0305-4470/34/23/308
http://arXiv.org e-Print archive, E-print No. quant-ph/9803084, March 1998.

[2] Bozhidar Z. Iliev. Fibre bundle formulation of nonrelativistic quantum mechanics. II. Equations of motion and observables. *Journal of Physics A: Mathematical and General*, 34(23):4919–4934, 2001. DOI No. 10.1088/0305-4470/34/23/309
http://arXiv.org e-Print archive, E-print No. quant-ph/9804062, April 1998.

[3] Bozhidar Z. Iliev. Fibre bundle formulation of nonrelativistic quantum mechanics. III. Pictures and integrals of motion. *Journal of Physics A: Mathematical and General*, 34(23):4935–4950, 2001. DOI No. 10.1088/0305-4470/34/23/310
http://arXiv.org e-Print archive, E-print No. quant-ph/9806046, June 1998.

[4] Bozhidar Z. Iliev. Fibre bundle formulation of nonrelativistic quantum mechanics. IV. Mixed states and evolution transport's curvature. *International Journal of Modern Physics A*, 17(2):229–243, 2002. DOI No. 10.1142/S0217751X02005669
http://arXiv.org e-Print archive, E-print No. quant-ph/9901039, January 1999.

[5] Bozhidar Z. Iliev. Fibre bundle formulation of nonrelativistic quantum mechanics. V. Interpretation, summary, and discussion. *International Journal of Modern Physics A*, 17(2):245–258, 2002. DOI No. 10.1142/S0217751X02005712
http://arXiv.org e-Print archive, E-print No. quant-ph/9902068, February 1999.

[6] Bozhidar Z. Iliev. Fibre bundle formulation of nonrelativistic quantum mechanics (full version).
http://arXiv.org e-Print archive, E-print No. quant-ph/0004041, April 2000.

[7] D. Husemoller. *Fibre bundles.* McGraw-Hill Book Co., New York-St. Louis-San Francisco-Toronto-London-Sydney, 1966. Other editions: Springer Verlag, 1975, 1994, 1998. Russian translation: Mir, Moscow, 1970.

[8] Norman Steenrod. *The topology of fibre bundles.* Princeton University Press, Princeton, ninth edition, 1974. First eddition: Princeton University Press, 1951.

[9] Serge Lang. *Differential manifolds.* Springer Verlag, New York, 1985. (originally published: Addison-Wesley Pub. Co, Reading, Mass. 1972).

[10] Bozhidar Z. Iliev. Transports along maps in fibre bundles. JINR Communication E5-97-2, Dubna, 1997. 19 pp.
http://arXiv.org e-Print archive, E-print No. dg-ga/9709016, September 1997.